Representing Elementary Semi-Algebraic Sets by a Few Polynomial Inequalities: A Constructive Approach

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Abstract

Let P be an elementary closed semi-algebraic set in $\mathbb{R}^d$, i.e., there exist real polynomials $p_1, \dots, p_s$ ($s \in \mathbb{N}$) such that $P = \{x \in \mathbb{R}^d : p_1(x) \geq 0, \dots, p_s(x) \geq 0\}$; in this case $p_1, \dots, p_s$ are said to represent P. Denote by n the maximal number of the polynomials from $\{p_1, \dots, p_s\}$ that vanish in a point of P. If P is non-empty and bounded, we show that it is possible to construct $n + 1$ polynomials representing P. Furthermore, the number $n + 1$ can be reduced to n in the case when the set of points of P in which n polynomials from $\{p_1, \dots, p_s\}$ vanish is finite. Analogous statements are also obtained for elementary open semi-algebraic sets.

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1 Introduction

In what follows $x := (x_1, \dots, x_d)$ is a variable vector in $\mathbb{R}^d$ ($d \in \mathbb{N}$). As usual, $\mathbb{R}[x] := \mathbb{R}[x_1, \dots, x_d]$ denotes the ring of polynomials in variables $x_1, \dots, x_d$ and coefficients in $\mathbb{R}$. A subset P of $\mathbb{R}^d$ which can be represented by

$$P = (p_1, \dots, p_s)_{\geq 0} := \{x \in \mathbb{R}^d : p_1(x) \geq 0, \dots, p_s(x) \geq 0\} \tag{1.1}$$

for $p_1, \dots, p_s \in \mathbb{R}[x]$ ($s \in \mathbb{N}$) is said to be an *elementary closed semi-algebraic set* in $\mathbb{R}^d$. Clearly, the number s from (1.1) is not uniquely determined by P. Let us denote by $s(d, P)$ the minimal s such that (1.1) is fulfilled for appropriate $p_1, \dots, p_s \in \mathbb{R}[x]$. Analogously, a subset P_0 of $\mathbb{R}^d$ which can be represented by

$$P_0 = (p_1, \dots, p_s)_{>0} := \{x \in \mathbb{R}^d : p_1(x) > 0, \dots, p_s(x) > 0\} \tag{1.2}$$

for some $p_1, \dots, p_s \in \mathbb{R}[x]$ ($s \in \mathbb{N}$) is said to be an *elementary open semi-algebraic set* in $\mathbb{R}^d$. The quantity $s_0(d, P_0)$ associated to P_0 is introduced analogously to $s(d, P)$. The system of polynomials $p_1, \dots, p_s$ from (1.1) (resp. (1.2)) is said to be a *polynomial representation* of P (resp. P_0). From the well-known Theorem of Bröcker and Scheiderer (see [ABR96, Chapter 5], and [BCR98, §6.5, §10.4] and the references therein) it follows that, for P and P_0 as above, the following inequalities are fulfilled:

$$s(d, P) \leq d(d+1)/2, \tag{1.3}$$

$$s_0(d, P_0) \leq d. \tag{1.4}$$

Both of these inequalities are sharp. It should be emphasized that all known proofs of (1.3) and (1.4) are highly non-constructive. The main aim of this paper is to provide constructive upper bounds for $s(d, P)$ and $s_0(d, P_0)$ for certain classes of P and P_0; see also [vH92], [Ber98], [GH03], [Hen07], [BGH05], and [AH07] for previous results on this topic. We also mention that constructive results on polynomial

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representations of special semi-algebraic sets are related to polynomial optimization; see [Las01], [Mar03], [Sch05], [Lau08], and [HN08].

Let $p_1, \dots, p_s \in \mathbb{R}[x]$ and let $P := (p_1, \dots, p_s)_{\geq 0}$ be non-empty. The assumptions of our main theorems are formulated in terms of the following functionals, which depend on $p_1, \dots, p_s$. The functional

$$I_x(p_1, \dots, p_s) := \big\{ i = 1, \dots, s \,:\, p_i(x) = 0 \big\}, \; x \in P, \tag{1.5}$$

determines the set of constraints defining P which are "active" in x. Furthermore, we define

$$n(p_1, \dots, p_s) \;:=\; \max \left\{ |I_x(p_1, \dots, p_s)| : x \in P \right\}, \tag{1.6}$$

$$X(p_1, \dots, p_s) \;:=\; \big\{ x \in P \,:\, |I_x(p_1, \dots, p_s)| = n(p_1, \dots, p_s) \big\}, \tag{1.7}$$

where $| \cdot |$ stands for the cardinality. The geometric meaning of $n(p_1, \dots, p_s)$ and $X(p_1, \dots, p_s)$ can be illustrated by the following special situation. Let P be a d-dimensional polytope with s facets (see [Zie95] for information on polytopes). Then P can be given by (1.1) with all p_i having degree one (the so-called *H-representation*). In this case $n(p_1, \dots, p_s)$ is the maximal number of facets of P having a common vertex and $X(p_1, \dots, p_s)$ is the set consisting of those vertices of P which are contained in the maximal number of facets of P. If the polytope P is *simple* (that is, each vertex of P lies in precisely d facets), then $n(p_1, \dots, p_s) = d$ and $X(p_1, \dots, p_s)$ is the set of all vertices of P.

Now we are ready to formulate our main results.

Theorem 1.1. *Let $p_1, \dots, p_s \in \mathbb{R}[x]$, $P := (p_1, \dots, p_s)_{\geq 0}$, and $P_0 := (p_1, \dots, p_s)_{>0}$. Assume that P is non-empty and bounded, and $n := n(p_1, \dots, p_s) < s$. Then the following inequalities are fulfilled:*

$$s(d, P) \leq n + 1, \qquad s_0(d, P_0) \leq n + 1$$

Furthermore, there exists an algorithm that gets $p_1, \dots, p_s$ and returns $n+1$ polynomials $q_0, \dots, q_n \in \mathbb{R}[x]$ satisfying $P = (q_0, \dots, q_n)_{\geq 0}$ and $P_0 = (q_0, \dots, q_n)_{>0}$. □

In the case when $X(p_1, \dots, p_s)$ is finite Theorem 1.1 can be improved.

Theorem 1.2. *Let $p_1, \dots, p_s \in \mathbb{R}[x]$, $P := (p_1, \dots, p_s)_{\geq 0}$, and $P_0 := (p_1, \dots, p_s)_{>0}$. Assume that P is non-empty and bounded, $X := X(p_1, \dots, p_s)$ is finite, and $n := n(p_1, \dots, p_s) < s$. Then the following inequalities are fulfilled:*

$$s(d, P) \leq n, \qquad s_0(d, P_0) \leq n$$

Furthermore, there exists an algorithm that gets $p_1, \dots, p_s$ and X and returns n polynomials $q_1, \dots, q_n$ satisfying $P = (q_1, \dots, q_n)_{\geq 0}$ and $P_0 = (q_1, \dots, q_n)_{>0}$. □

Below we discuss existing results and problems related to Theorems 1.1 and 1.2. Let P be a convex polygon in $\mathbb{R}^2$ with s edges, which is given by (1.1) with all p_i having degree one. Bernig [Ber98] showed that setting $q_2 := p_1 \cdot \ldots \cdot p_s$ one can construct a strictly concave polynomial $q_1(x)$ vanishing on all vertices of P which satisfies $P = (q_1, q_2)_{\geq 0}$; see Fig. 1. As it will be seen from the proof of Theorem 1.2, for the case $d = 2$ and P as in Theorem 1.2 we also set $q_2 := p_1 \cdot \ldots \cdot p_s$ and choose q_1 in such a way that it vanishes on each point of X and the set $(q_1)_{\geq 0}$ approximates P sufficiently well; see Fig. 2. However, since P from Theorem 1.2 is in general not convex, the construction of q_1 requires a different idea. The statement of Theorem 1.2 concerned with P_0 and restricted to the cases $n = 2$ and $n = d$, $s = d+1$ (with slightly different assumptions on P_0) was obtained by Bernig [Ber98, Theorems 4.1.1 and 4.3.5].

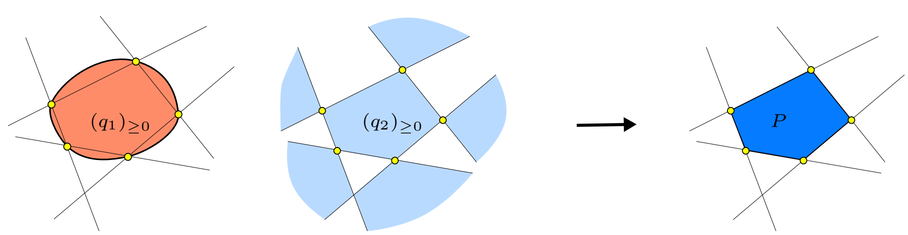

Figure 1. Illustration to the result of Bernig on convex polygons

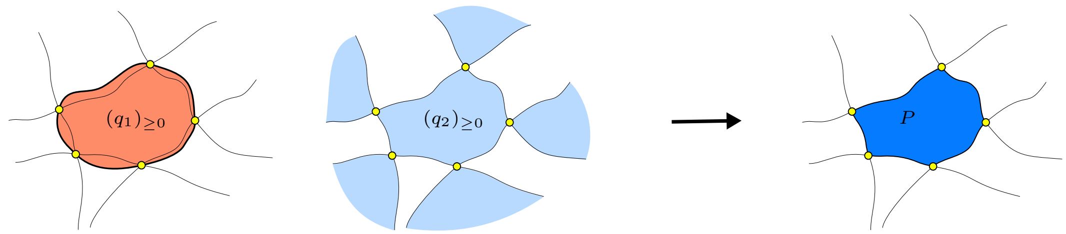

Figure 2. Illustration to Theorem 1.2 for the case $d = 2$, $n = 2$

The study of $s(d, P)$ for the case when P is a polyhedron of an arbitrary dimension was initiated by Grötschel and Henk [GH03]. In [GH03, Corollary 2.2(i)] it was noticed that $s(d, P) \geq d$ for every d-dimensional polytope P. On the other hand, Bosse, Grötschel, and Henk [BGH05] gave an upper bound for $s(d, P)$ which is linear in d for the case of an arbitrary d-dimensional polyhedron P. In particular, they showed that $s(d, P) \leq 2d - 1$ if P is d-dimensional polytope. In [BGH05] the following conjecture was announced.

Conjecture 1.3. (Bosse & Grötschel & Henk 2005) *For every d-dimensional polytope P in $\mathbb{R}^d$ the equality $s(d, P) = d$ holds.* □

This conjecture has recently been confirmed for all simple d-dimensional polytopes; see [AH07].

Theorem 1.4. (Averkov & Henk 2007+) *Let P be a d-dimensional simple polytope Then $s(d, P) = d$. Furthermore, there exists an algorithm that gets polynomials $p_1, \ldots, p_s$ ($s \in \mathbb{N}$) of degree one satisfying $P = (p_1, \ldots, p_s)_{\geq 0}$ and returns d polynomials $q_1, \ldots, q_d$ satisfying $P = (q_1, \ldots, q_d)_{\geq 0}$.* □

Elementary closed semi-algebraic sets $P := (p_1, \ldots, p_s)_{\geq 0}$ with $n(p_1, \ldots, p_s) = d$ can be viewed as natural extensions of simple polytopes in the framework of real algebraic geometry. Thus, we can see that Theorem 1.4 is a consequence of Theorem 1.2. Fig. 3 illustrates Theorem 1.4 for the case when P is a three-dimensional cube. This figure can also serve as an illustration of Theorem 1.2 with the only difference that in Theorem 1.2 the set $(p_1)_{\geq 0}$ does not have to be convex anymore.

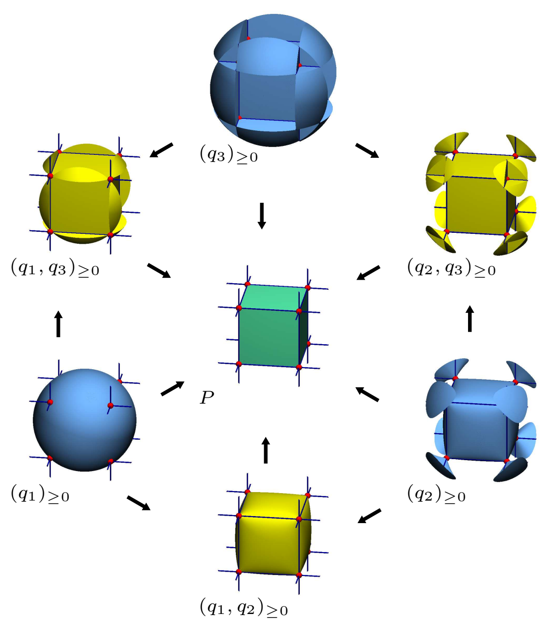

Figure 3. Illustration to Theorem 1.4 (and Theorem 1.2) for the case when P is a three-dimensional cube.

While proving our main theorems we derive the following approximation results which can be of independent interest. The *Hausdorff distance* δ is a metric defined on the space of non-empty compact subsets of $\mathbb{R}^d$ by the equality

$$\delta(A,B) := \max\Big\{\max_{a\in A}\min_{b\in B}\|a-b\|, \max_{b\in B}\min_{a\in A}\|a-b\|\Big\},$$

see [Sch93, p. 48].

Theorem 1.5. *Let $p_1,\dots,p_s \in \mathbb{R}[x]$, $P := (p_1,\dots,p_s)_{\geq 0}$, and $P_0 := (p_1,\dots,p_s)_{>0}$. Assume that P is non-empty and bounded. Then there exists an algorithm that gets $p_1,\dots,p_s$ and $\varepsilon > 0$ and returns a polynomial $q \in \mathbb{R}[x]$ such that $P_0 \subseteq (q)_{>0}$, $P \subseteq (q)_{\geq 0}$, and the Hausdorff distance from P to $(q)_{\geq 0}$ is at most ε.* □

Theorem 1.6. *Let $p_1,\dots,p_s \in \mathbb{R}[x]$, $P := (p_1,\dots,p_s)_{\geq 0}$, and $P_0 := (p_1,\dots,p_s)_{>0}$. Assume that P is non-empty and bounded, $X := X(p_1,\dots,p_s)$ is finite, and $n := n(p_1,\dots,p_s) < s$. Then there exists an algorithm that gets $p_1,\dots,p_s$, X, and $\varepsilon > 0$ and returns a polynomial $q \in \mathbb{R}[x]$ such that $P_0 \subseteq (q)_{>0}$, $P \subseteq (q)_{\geq 0}$, the Hausdorff distance from P to $(q)_{\geq 0}$ is at most ε, and $q(x) = 0$ for every $x \in X$.* □

We note that some further results on approximation by sublevel sets of polynomials can be found in [Ham63], [Fir74], and [GH03, Lemma 2.6].

The paper has the following structure. Section 2 contains preliminaries from real algebraic geometry. In Section 3 we obtain approximation results (including Theorems 1.5 and 1.6). Finally, in Section 4 the proofs of Theorems 1.1 and 1.2 are presented. In the beginning of the proofs of Theorems 1.1 and 1.2 one can find the formulas defining the polynomials q_i (see (4.2) and (4.3)) as well as sketches of the main arguments.

2 Preliminaries from real algebraic geometry

The origin and the Euclidean norm in $\mathbb{R}^d$ are denoted by o and $\|\cdot\|$, respectively. We endow $\mathbb{R}^d$ with its Euclidean topology. By $B^d(c,\rho)$ we denote the closed Euclidean ball in $\mathbb{R}^d$ with center at $c \in \mathbb{R}^d$ and radius $\rho > 0$. The interior (of a set) is abbreviated by int. We also define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ is the set of all natural numbers.

A set $A \subseteq \mathbb{R}^d$ given by

$$A := \bigcup_{i=1}^{k} \left\{x \in \mathbb{R}^d : f_{i,1}(x) > 0, \dots, f_{i,s_i}(x) > 0,\ g_i(x) = 0\right\},$$

where $i \in \{1,\dots,k\}$, $j \in \{1,\dots,s_i\}$ and $f_{i,j}, g_i \in \mathbb{R}[x]$, is called *semi-algebraic*.

An expression Φ is called a *first-order formula over the language of ordered fields with coefficients in* $\mathbb{R}$ if Φ is a formula built with a finite number of conjunctions, disjunctions, negations, and universal or existential quantifier on variables, starting from formulas of the form $f(x_1,\dots,x_d) = 0$ or $g(x_1,\dots,x_d) > 0$ with $f, g \in \mathbb{R}[x]$; see [BCR98, Definition 2.2.3]. The *free variables* of Φ are those variables, which are not quantified. A formula with no free variables is called a *sentence*. Each sentence is is either true or false. The following proposition is well-known; see also [BCR98, Proposition 2.2.4] and [BPR06, Corollary 2.75].

Proposition 2.1. *Let Φ be a first-order formula over the language of ordered fields with coefficients in $\mathbb{R}$ and free variables $y_1,\dots,y_m$. Then the set*

$$\{(y_1,\dots,y_m) \in \mathbb{R}^m : \Phi(y_1,\dots,y_m)\},$$

consisting of all $(y_1,\dots,y_m) \in \mathbb{R}^d$ for which Φ is true, is semi-algebraic. □

A real valued function $f(x)$ defined on a semi-algebraic set A is said to be a *semi-algebraic function* if its graph is a semi-algebraic set in $\mathbb{R}^{d+1}$. The following theorem presents *Łojasiewicz's Inequality*; see [Łoj59] and [BCR98, Corollary 2.6.7].

Theorem 2.2. (Łojasiewicz 1959) *Let A be non-empty, bounded, and closed semi-algebraic set in $\mathbb{R}^d$. Let f and g be continuous, semi-algebraic functions defined on A and such that $\{x \in A : f(x) = 0\} \subseteq \{x \in A : g(x) = 0\}$. Then there exist $M \in \mathbb{N}$ and $\lambda \geq 0$ such that*

$$|g(x)|^M \leq \lambda |f(x)|$$

for every $x \in A$. □

Considering algorithmic questions we use the following standard settings; see [ABR96, Chapter §8.1]. It is assumed that a polynomial in $\mathbb{R}[x]$ is given by its coefficients and that a finite list of real coefficients occupies finite memory space. Furthermore, arithmetic and comparison operations over reals are assumed to be atomic, i.e., computable in one step. The following well-known result is relevant for the constructive part of our theorems; see [BPR06, Algorithm 12.30].

Theorem 2.3. (Tarski 1951, Seidenberg 1954) *Let Φ be a sentence over the language of ordered fields with coefficients in $\mathbb{R}$. Then there exists an algorithm that gets Φ and decides whether Φ is true or false.* □

3 Approximation results

The following proposition (see [Sch93, p. 57]) presents a characterization of the convergence with respect to the Hausdorff distance.

Proposition 3.1. *A sequence $(A_n)_{n=1}^{+\infty}$ of compact convex sets in $\mathbb{R}^d$ converges to a compact set A in the Hausdorff distance if and only if the following conditions are fulfilled:*

1. *Every point of A is a limit of a sequence $(a_k)_{k=1}^{+\infty}$ satisfying $a_k \in A_k$ for every $k \in \mathbb{N}$.*
2. *If $(k_j)_{j=1}^{+\infty}$ is a strictly increasing sequence of natural numbers and $(a_{k_j})_{j=1}^{+\infty}$ is a convergent sequence satisfying $a_{k_j} \in A_{k_j}$ $(j \in \mathbb{N})$, then a_{k_j} converges to a point of A, as $j \to +\infty$.*
3. *The set $\bigcup_{k=1}^{+\infty} A_k$ is bounded.*

□

Let $p_1, \ldots, p_s \in \mathbb{R}[x]$. The following theorem states that for the case when $P := (p_1, \ldots, p_s)_{\geq 0}$ is non-empty and bounded, appropriately relaxing the inequalities $p_i(x) \geq 0$, which define P, we get a bounded semi-algebraic set that approximates P arbitrarily well. Let us define

$$P(M, \varepsilon) := \left\{x \in \mathbb{R}^d : (1 + \|x\|^2)^M p_i(x) \geq -\varepsilon \text{ for } 1 \leq i \leq s\right\} \tag{3.1}$$

with $M \in \mathbb{N}_0$ and $\varepsilon > 0$.

Theorem 3.2. *Let $p_1, \ldots, p_s \in \mathbb{R}[x]$, $P := (p_1, \ldots, p_s)_{\geq 0}$, and $P_0 := (p_1, \ldots, p_s)_{>0}$. Assume that P is non-empty and bounded. Then there exists an algorithm that gets $p_1, \ldots, p_s$ and returns values $M \in \mathbb{N}_0$ and $\varepsilon_0 > 0$ such that the following conditions are fulfilled:*

1. *$P(M, \varepsilon)$ is bounded for $\varepsilon = \varepsilon_0$.*
2. *$P(M, \varepsilon)$, $\varepsilon \in (0, \varepsilon_0]$, converges to P in the Hausdorff distance, as $\varepsilon \to 0$.*

□

Proof. First we show the existence of M and ε_0 from the assertion, and after this we show that these two quantities are constructible. Let us derive the existence of M and ε_0 satisfying Condition 1. Since P is bounded, after replacing P by an appropriate homothetical copy, we may assume that $P \subseteq \operatorname{int} B^d(o, 1)$. By Proposition 2.1, the function

$$f(x) := -\min_{1 \leq i \leq s} p_i(x)$$

is semi-algebraic. We also have $f(x) > 0$ for all $x \in \mathbb{R}^d$ with $\|x\| \geq 1$. Furthermore, the set $P(M, \varepsilon)$ can be expressed with the help of $f(x)$ by

$$P(M, \varepsilon) = \left\{x \in \mathbb{R}^d : (1 + \|x\|^2)^M f(x) \leq \varepsilon\right\}. \tag{3.2}$$

For $t \geq 1$ the function

$$a(t) := \min \{f(x) : 1 \leq \|x\| \leq t\}$$

is positive and non-increasing. Using Proposition 3.1 it can be shown that $a(t)$ is continuous. Moreover, in view of Proposition 2.1, we see that $a(t)$ is semi-algebraic. In the case $\inf \{a(t) : t \geq 1\} > 0$ Condition 1 is fulfilled for $M = 0$ and $\varepsilon_0 = \frac{1}{2} \inf \{a(t) : t \geq 1\}$. In the opposite case we have $a(t) \to 0$, as $t \to +\infty$. Then

$$b(t) := \begin{cases} a(1/t), & 0 < t \leq 1, \\ 0, & t = 0 \end{cases}$$

is a continuous semi-algebraic function on $[0, 1]$ with $b(t) = 0$ if and only if $t = 0$. Thus, applying Theorem 2.2 to the functions $b(t)$ and t^2 defined on $[0, 1]$, we see that there exist $M \in \mathbb{N}_0$ and $\gamma > 0$ such that $t^{2M} \leq \gamma\, b(t)$ for every $t \in [0, 1]$. Consequently $t^{2M} a(t) \geq \frac{1}{\gamma}$ for every $t \geq 1$. The latter implies that $(1 + \|x\|^2)^M f(x) \geq \frac{1}{\gamma}$, and Condition 1 is fulfilled for M as above and $\varepsilon_0 = \frac{1}{2\gamma}$. Now we show that Condition 1 implies Condition 2. Assume that Condition 1 is fulfilled. Then the set $P(M, \varepsilon)$ is bounded for all $\varepsilon \in (0, \varepsilon_0]$. Hence $\delta(P, P(M, \varepsilon))$ is well defined for all $\varepsilon \in (0, \varepsilon_0]$. Consider an arbitrary sequence $(t_j)_{j=1}^{+\infty}$ with $t_j \in (0, \varepsilon_0]$ and $t_j \to 0$, as $j \to +\infty$, using Proposition 3.1 we can see that $\delta(P, P(t_j)) \to 0$, as $j \to +\infty$. Consequently, Condition 2 is fulfilled.

Finally we show that ε_0 and M are constructible. For determination of M one can use the following "brute force" procedure.

Procedure: Determination of M.

Input: $p_1, \dots, p_s \in \mathbb{R}[x]$.

Output: A number $M \in \mathbb{N}_0$ such that for some $\varepsilon_0 > 0$ the set $P(M, \varepsilon_0)$ is bounded.

1: Set $M := 0$.

2: For $i \in \{1, \dots, s\}$ introduce the first-order formula

$$\Phi_i := \text{"}(1 + x_1^2 + \cdots + x_d^2)^M\, p_i(x_1, \dots, x_d) \geq -\varepsilon_0\text{"}$$

with free variables $x_1, \dots, x_d, \varepsilon_0$.

3: Test the existence of $\varepsilon_0 > 0$ for which $P(M, \varepsilon_0)$ is bounded. More precisely, determine whether the sentence

$$\Psi := \text{"}(\exists \varepsilon_0)(\exists \tau) \quad (\varepsilon_0 > 0) \wedge (\forall x_1) \dots (\forall x_d) \left(\Phi_1 \wedge \ldots \wedge \Phi_s \to (x_1^2 + \cdots + x_d^2 \leq \tau^2)\right)\text{"}$$

is true or false (cf. Theorem 2.3).

4: If Ψ is true, return M and stop. Otherwise set $M := M + 1$ and go to Step 2.

In view of the conclusions made in the proof, the above procedure terminates after a finite number of iterations. For determination of ε_0 we can use a similar procedure. We start with $\varepsilon_0 := 1$ and assign $\varepsilon_0 := \varepsilon_0/2$ at each new iteration, terminating the cycle as long as $P(M, \varepsilon_0)$ is bounded. $\square$

Remark 3.3. We wish to show Theorem 3.2 cannot be improved by setting $M := 0$, since $P(0, \varepsilon)$ may be unbounded for all $\varepsilon > 0$. Let us consider the following example. Let $M = 0$, $d = 2$, $s = 1$, and

$$p_1(x) = -(x_1 - x_2)^2 - (x_1^2 + x_2^2 - 1)\,(1 + x_1^2 - x_2^2)^2.$$

Then the set $P = (p_1)_{\geq 0}$ is bounded. In fact, if $\|x\| > 1$, then the term $x_1^2 + x_2^2 - 1$, appearing in the definition of p_1, is positive. But the remaining terms $x_1 - x_2$ and $1 + x_1^2 - x_2^2$ cannot vanish simultaneously. Hence, $p_1(x) < 0$ for every x with $\|x\| > 1$, which shows that $P \subseteq B^2(o, 1)$. Furthermore, since $p_1(o) < 0$, we see that P has non-empty interior (which shows that our example is non-degenerate enough). Let us show that $P(M, \varepsilon) = \left\{x \in \mathbb{R}^2 : q_1(x) \geq -\varepsilon\right\}$ is unbounded for every $\varepsilon > 0$. For $x(t) := (t, \sqrt{1 + t^2})$ with $t \geq 0$ one has $\|x(t)\| = \sqrt{1 + 2t^2} \to +\infty$ and $p_1(x(t)) = -\left(t - \sqrt{1 + t^2}\right)^2 \to 0^-$, as $t \to +\infty$; see also Fig. 4. This implies unboundedness of $P(M, \varepsilon)$. $\square$

Throughout the rest of the paper we shall use the following polynomials associated to $p_1, \dots, p_s \in \mathbb{R}[x]$. For $M \in \mathbb{N}_0$, $\lambda > 0$, and $k \in \mathbb{N}$ we define

$$g_{M,\lambda,k}(x) := \frac{1}{s} \sum_{i=1}^{s} \Big(1 - \frac{1}{\lambda}(1 + \|x\|^2)^M \, p_i(x)\Big)^{2k} \tag{3.3}$$

If $X := X(p_1, \dots, p_s)$ is finite, we define

$$h_\mu(x) := \prod_{v \in X} \left(\frac{\|x - v\|}{\mu}\right)^2,$$

where $\mu > 0$.

Lemma 3.4. *Let $p_1, \dots, p_s \in \mathbb{R}[x]$, $P := (p_1, \dots, p_s)_{\geq 0}$, and $P_0 := (p_1, \dots, p_s)_{>0}$. Assume that P is non-empty and bounded. Then for every $\varepsilon > 0$, $M \in \mathbb{N}_0$, $\lambda > 0$, and $k \in \mathbb{N}$ satisfying*

$$\begin{aligned} \lambda &\geq \max_{1 \leq i \leq s} \max_{x \in P} (1 + \|x\|^2)^M p_i(x), && (3.4)\\ s &\leq \Big(1 + \frac{\varepsilon}{\lambda}\Big)^{2k} && (3.5) \end{aligned}$$

the polynomial $g(x) := g_{M,\lambda,k}(x)$ fulfills the relations

$$\begin{aligned} P_0 &\subseteq \big\{x \in \mathbb{R}^d : g(x) < 1\big\} \subseteq P(M, \varepsilon), && (3.6)\\ P &\subseteq \big\{x \in \mathbb{R}^d : g(x) \leq 1\big\} \subseteq P(M, \varepsilon). && (3.7) \end{aligned}$$

Furthermore, there exists an algorithm that gets $p_1, \dots, p_s$, $\varepsilon > 0$, and $M \in \mathbb{N}_0$ and constructs $g = g_{M,\lambda,k} \in \mathbb{R}[x]$ satisfying (3.6) and (3.7).

Proof. Inclusions $P_0 \subseteq \big\{x \in \mathbb{R}^d : g(x) < 1\big\}$ and $P \subseteq \big\{x \in \mathbb{R}^d : g(x) \leq 1\big\}$ follow from (3.4). It remains to show the inclusion $\big\{x \in \mathbb{R}^d : g(x) \leq 1\big\} \subseteq P(M, \varepsilon)$. Assume that $g(x) \leq 1$. Then

$$\max_{1 \leq i \leq s} \Big(1 - \frac{1}{\lambda}(1 + \|x\|^2)^M \, p_i(x)\Big)^{2k} \leq s \overset{(3.5)}{\leq} \Big(1 + \frac{\varepsilon}{\lambda}\Big)^{2k}.$$

Consequently

$$\max_{1 \leq i \leq s} \Big(1 - \frac{1}{\lambda}(1 + \|x\|^2)^M \, p_i(x)\Big) \leq 1 + \frac{\varepsilon}{\lambda},$$

or equivalently, $(1 + \|x\|^2)^M f(x) \leq \varepsilon$. Hence $x \in P(M, \varepsilon)$.

Now let us discuss the constructibility of $g(x)$. It suffices to show the constructibility of λ satisfying (3.4). For determination of λ we iterate starting with $\lambda := 1$, set $\lambda := \lambda + 1$ at each new step, and use (3.4), reformulated as a first-order formula, as a condition for terminating the cycle. $\square$

One can see that Theorem 1.5 from the introduction is a direct consequence of Theorem 3.2 and Lemma 3.4.

Theorem 3.5. *Let $p_1, \dots, p_s \in \mathbb{R}^d$, $P := (p_1, \dots, p_s)_{\geq 0}$, and $P_0 := (p_1, \dots, p_s)_{>0}$. Assume that P is non-empty and bounded, $X := X(p_1, \dots, p_s)$ is finite, and $n := n(p_1, \dots, p_s) < s$. Then there exists an algorithm that gets $p_1, \dots, p_s$, X, $M \in \mathbb{N}_0$, and $\varepsilon > 0$ and returns $q \in \mathbb{R}[x]$ fulfilling the relations*

$$\begin{array}{rcccl} P_0 & \subseteq & (q)_{>0} & \subseteq & P(M, 2\varepsilon), \\ P & \subseteq & (q)_{\geq 0} & \subseteq & P(M, 2\varepsilon), \\ & & X & \subseteq & \big\{x \in \mathbb{R}^d : q(x) = 0\big\}. \end{array}$$

Furthermore, q can be defined by

$$q(x) := \sigma_{s-n+1}(p_1(x), \dots, p_s(x)) - g_{M,\lambda,k}(x)^l h_\mu(x)^m,$$

where $k, l, m \in \mathbb{N}$, $\lambda > 0$, and $\mu > 0$. $\square$

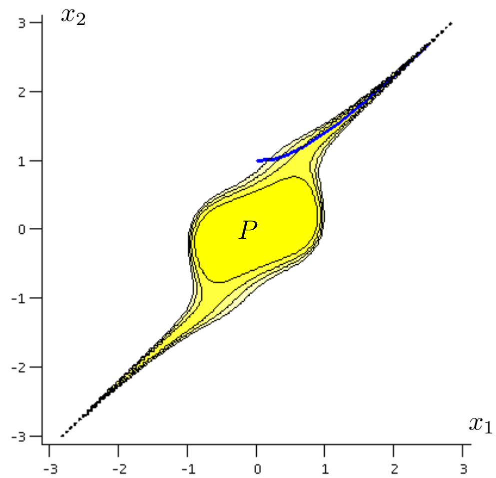

Figure 4. Illustration to Remark 3.3: the level sets given by equations $p_1(x) = 0$, $p_1(x) = -0.3$, $p_1(x) = -0.5$, $p_1(x) = -0.7$ and a part of the curve with parametrization $x(t)$

Proof. Analogously to the proof of Theorem 3.2, we first show the existence of q from the assertion and then we derive the constructive part of the theorem. We fix λ and k satisfying (3.4) and (3.5) and set $g(x) := g_{M,\lambda,k}(x)$. Let us derive the inclusions $P_0 \subseteq (q)_{>0}$ and $P \subseteq (q)_{\geq 0}$. First we show that

$$\max_{x \in P} g(x) < 1. \tag{3.8}$$

Let $I_x := I_x(p_1, \ldots, p_s)$. Since $n < s$, for every $x \in P$ the set I_x is properly contained in $\{1, \ldots, s\}$. Consequently, for every $x \in P$ we get

$$g(x) = \frac{1}{s} \left(|I_x| + \sum_{i \in \{1,\ldots,s\} \setminus I_x} \left(1 - \frac{1}{\lambda} (1 + \|x\|^2)^M p_i(x) \right)^{2k} \right) < 1.$$

Thus, (3.8) is fulfilled. Therefore we can fix α with

$$\max_{x \in P} g(x) \leq \alpha < 1. \tag{3.9}$$

In view of (3.9) and the finiteness of X, we can fix $\rho > 0$ such that

$$\bigcup_{v \in X} B^d(v, \rho) \subseteq \left\{ x \in \mathbb{R}^d : g(x) \leq 1 \right\}. \tag{3.10}$$

and

$$B^d(v, \rho) \cap B^d(w, \rho) = \emptyset \tag{3.11}$$

for all $v, w \in X$ with $v \neq w$.

Let us consider an arbitrary $x \in P$. We show that, for an appropriate choice of $l \in \mathbb{N}$ and $m \in \mathbb{N}$ we have $q(x) \geq 0$, and the latter inequality is strict for $x \in P_0$.

Case A: $x \in P \cap \left(\bigcup_{v \in X} B^d(v, \rho) \right)$. Let us fix $w \in X$ such that $\|x - w\| \leq \rho$. Since $x \in P$, we have $\sigma_{s-n+1}(p_1(x), \ldots, p_s(x)) \geq 0$. Furthermore, due to the choice of ρ, equality is attained if and only if $x = w$. Let $\mu > 0$ be an arbitrary scalar satisfying

$$\mu \geq \operatorname{diam}(P) := \max \left\{ \|x' - x''\| : x', x'' \in P \right\}. \tag{3.12}$$

Applying Theorem 2.2 to the functions $\sigma_{s-n+1}(p_1(x), \ldots, p_s(x))$ and $\left(\frac{\|x-w\|}{\mu} \right)^2$ restricted to $B^d(w, \rho) \cap P$, we have

$$\left(\frac{\|x - w\|}{\mu} \right)^{2m(w)} \leq \tau(w) \cdot \sigma_{s-n+1}(p_1(x), \ldots, p_s(x))$$

for appropriate parameters $\tau(w) > 0$ and $m(w) \in \mathbb{N}$ independent of x. In view of the choice of μ we deduce

$$\left(\frac{\|x-w\|}{\mu}\right)^{2m} \leq \tau \cdot \sigma_{s-n+1}(p_1(x), \ldots, p_s(x)), \tag{3.13}$$

where $\tau := \max_{v \in X} \tau(v)$ and $m := \max_{v \in X} m(v)$. We have

$$\begin{aligned} g(x)^l h_\mu(x)^m &\overset{(3.9)}{\leq} \alpha^l \, h_\mu(x)^m = \alpha^l \, \left(\frac{\|x-w\|}{\mu}\right)^{2m} \prod_{v \in X \setminus \{w\}} \left(\frac{\|x-v\|}{\mu}\right)^{2m} \overset{(3.12)}{\leq} \alpha^l \left(\frac{\|x-w\|}{\mu}\right)^{2m} \\ &\overset{(3.13)}{\leq} \tau \, \alpha^l \, \sigma_{s-n+1}(p_1(x), \ldots, p_s(x)). \end{aligned} \tag{3.14}$$

In view of (3.9), for all sufficiently large $l \in \mathbb{N}$ the inequality

$$\tau \, \alpha^l < 1, \tag{3.15}$$

is fulfilled. Assuming that (3.15) holds, and taking into account (3.14), we have $q(x) \geq 0$.

Now assume that x lies in $P_0 \cap \left(\bigcup_{v \in X} B^d(v, \rho)\right)$. Then, if l satisfies (3.15), we get $q(x) > 0$.

Case B: $x \in P \setminus \bigcup_{v \in X} B^d(v, \rho)$. Then $\|x - v\| \geq \rho$ for every $v \in X$. From the definition of elementary symmetric functions and the assumptions it easily follows that

$$\min\Big\{\sigma_{s-n+1}(p_1(x'), \ldots, p_s(x')) \, : \, x' \in P \setminus \bigcup_{v \in X} \operatorname{int} B^d(v, \rho)\Big\} > 0.$$

Let us choose γ with

$$0 < \gamma \leq \min\Big\{\sigma_{s-n+1}(p_1(x'), \ldots, p_s(x')) \, : \, x' \in P \setminus \bigcup_{v \in X} \operatorname{int} B^d(v, \rho)\Big\}. \tag{3.16}$$

Thus, we get the bounds

$$g(x)^l h_\mu(x)^m \overset{(3.9)}{\leq} \alpha^l h_\mu(x)^m \overset{(3.12)}{\leq} \alpha^l$$

and $\gamma \leq \sigma_{s-n+1}(p_1(x), \ldots, p_s(x))$. In view of (3.9), for all sufficiently large $l \in \mathbb{N}$ the inequality

$$\alpha^l < \gamma \tag{3.17}$$

is fulfilled. Assuming that (3.17) is fulfilled, we obtain $q(x) > 0$.

Now we show the inclusion $(q)_{\geq 0} \subseteq P(M, 2\varepsilon)$. Consider an arbitrary $x \in \mathbb{R}^d \setminus P(M, 2\varepsilon)$. Then

$$\min_{1 \leq i \leq s} (1 + \|x\|^2)^M \, p_i(x) \leq -2 \, \varepsilon,$$

which is equivalent to

$$\max_{1 \leq i \leq s} \left(1 - \frac{1}{\lambda}(1 + \|x\|^2)^M \, p_i(x)\right) \geq 1 + \frac{2\varepsilon}{\lambda}. \tag{3.18}$$

The latter implies that

$$\sum_{i=1}^{s} \Big(1 - \frac{1}{\lambda}(1 + \|x\|^2)^M \, p_i(x)\Big)^{2k} \geq \left(1 + \frac{2\varepsilon}{\lambda}\right)^{2k},$$

and therefore

$$g(x) \geq \frac{1}{s} \left(1 + \frac{2\varepsilon}{\lambda}\right)^{2k} \overset{(3.5)}{\geq} \left(\frac{\lambda + 2\varepsilon}{\lambda + \varepsilon}\right)^{2k} > 1. \tag{3.19}$$

We have

$$
\begin{aligned}
\left|\sigma_{s-n+1}(p_1(x),\dots,p_s(x))\right| &\le \sigma_{s-n+1}(|p_1(x)|,\dots,|p_s(x)|)\\
&\le \sigma_{s-n+1}(\underbrace{1,\dots,1}_{s}) \max_{1\le j\le s} |p_j(x)|^{s-n+1}\\
&= \tbinom{s}{n-1} \max_{1\le j\le s} |p_j(x)|^{s-n+1}\\
&\le \tbinom{s}{n-1} \lambda^{s-n+1} \max_{1\le j\le s} \left|\tfrac{1}{\lambda}\,(1+\|x\|^2)^M\, p_j(x)\right|^{s-n+1}\\
&\le \tbinom{s}{n-1} \lambda^{s-n+1} \left(\max_{1\le j\le s} \left|1-\tfrac{1}{\lambda}(1+\|x\|^2)^M p_j(x)\right| + 1\right)^{s-n+1}\\
&\overset{(3.18)}{\le} \tbinom{s}{n-1} \lambda^{s-n+1} \left(\max_{1\le j\le s} \left|1-\tfrac{1}{\lambda}(1+\|x\|^2)^M p_j(x)\right|^{2k} + 1\right)^{s-n+1}\\
&\overset{(3.19)}{\le} \tbinom{s}{n-1} \lambda^{s-n+1} \big(s\, g(x)+1\big)^{s-n+1}\\
&\overset{(3.19)}{\le} \tbinom{s}{n-1} \lambda^{s-n+1} \big(s\, g(x)+g(x)\big)^{s-n+1}\\
&= \tbinom{s}{n-1} \lambda^{s-n+1} (s+1)^{s-n+1} g(x)^{s-n+1}.
\end{aligned}
$$

The above estimate for $|\sigma_{s-n+1}(p_1(x),\dots,p_s(x))|$ together with the estimate

$$
h_\mu(x)^m = \prod_{v\in X} \left(\frac{\|x-v\|}{\mu}\right)^{2m} \ge \left(\frac{\rho}{\mu}\right)^{2m\,|X|} > 0
$$

and (3.19) implies that $|\sigma_{s-n+1}(p_1(x),\dots,p_s(x))| \le \frac{1}{2} g(x)^l h_\mu(x)^m$ if l fulfills the inequality

$$
2\binom{s}{n-1} \lambda^{s-n+1}(s+1)^{s-n+1} \le \left(\frac{\lambda+2\varepsilon}{\lambda+\varepsilon}\right)^{2k\,(l-s+n-1)} \left(\frac{\rho}{\mu}\right)^{2m|X|}. \tag{3.20}
$$

Since $\frac{\lambda+2\varepsilon}{\lambda+\varepsilon} > 1$, (3.20) is fulfilled if $l \in \mathbb{N}$ is large enough. Thus, we obtain that the inequality $q(x) < 0$ holds for all sufficiently large l.

Now we show the constructive part of the assertion. We present a sketch of a possible procedure that determines q. It suffices to evaluate the parameters k, l, m, λ, and μ involved in the definition of q. Constructibility of λ and k follows from Lemma 3.4. Let us apply Theorem 2.3 in the same way as in the previous proofs. Determine the following parameters in the given sequence. We can determine m satisfying (3.13) for an appropriate $\tau > 0$ and all $x \in P \cap \left(\bigcup_{v\in X} B^d(v,\rho)\right)$ using the same idea as in the procedure for determination of M in the proof of Theorem 3.2. A parameter μ satisfying (3.12) is constructible in view of Theorem 2.3 (by means of iteration procedure which we also used in the previous proofs). An appropriate l can be easily found from inequalities (3.15), (3.17), and (3.20). Thus, for evaluation of l we should first find the parameters τ, α, and ρ appearing in (3.15), (3.17), and (3.20). The parameters α, τ, and γ are determined by means of (3.9), (3.13), and (3.16). □

One can see that Theorem 1.6 from the introduction is a straightforward consequence of Theorem 3.2 and Theorem 3.5.

Remark 3.6. The parameters k, l, m, M, λ, μ involved in the statements of this section were computed with the help of the Theorem 2.3. In contrast to this, in general it is not possible to compute X exactly, since evaluation of X would involve solving a polynomial system of equations. This explains why in the statement of Theorem 3.5 the set X is taken as a part of the input. □

Remark 3.7. The parameters λ and μ from Lemma 3.4 and Theorem 3.5, respectively, are upper bounds for certain polynomial programs. In fact, by (3.4) the parameter $\lambda > 0$ is a common upper bound for the optimal solutions of s non-linear programs $p_i(x) \to \max,\ i \in \{1,\dots,s\}$, with constraints $p_j(x) \ge 0$, $1 \le j \le s$. From the proof of Theorem 3.5 we see that μ can be any number satisfying $\mu \ge \operatorname{diam}(P)$. Hence μ^2 is an upper bound for the optimal solution of the polynomial program $\|x'-x''\|^2 \to \max,\ x', x'' \in \mathbb{R}^d$, with $2d$ unknowns (which are coordinates of x' and x'') and the $2s$ constraints $p_i(x') \ge 0$ and $p_i(x'') \ge 0$, $1 \le i \le s$. The same observations apply also to the parameters α and γ from the proof of Theorem 3.5, which are used for determination of l. In this respect we notice that upper bounds of polynomial programs can be determined using convex relaxation methods; see [Las01], [Mar03], and [Sch05]. □

4 Proofs of the main theorems

Given $s \in \mathbb{N}$, $k \in \{1, \ldots, s\}$, and $y := (y_1, \ldots, y_s) \in \mathbb{R}^s$ the *k-th elementary symmetric function* in variables $y_1, \ldots, y_s$ is defined by

$$\sigma_k(y) := \sum_{\substack{I \subseteq \{1,\ldots,s\} \\ |I|=k}} \prod_{i \in I} y_i. \tag{4.1}$$

We also put $\sigma_0(y) := 1$.

Proposition 4.1. (Bernig 1998) *Let $y := (y_1, \ldots, y_s) \in \mathbb{R}^s$ with $s \in \mathbb{N}$. Then the following statements hold:*

I. $y_1 \geq 0, \ldots, y_s \geq 0$ if and only if $\sigma_1(y) \geq 0, \ldots, \sigma_s(y) \geq 0$.

II. $y_1 > 0, \ldots, y_s > 0$ if and only if $\sigma_1(y) > 0, \ldots, \sigma_s(y) > 0$.

Proof. The necessities of both of the parts are trivial. Let us prove the sufficiencies. We introduce the polynomial $f(t) = (t+y_1) \cdot \ldots \cdot (t+y_s)$, whose roots are the the values $-y_1, \ldots -y_s$. By *Vieta's formulas*, we have $f(t) = \sigma_s(y)\, t^0 + \sigma_{s-1}(y)\, t^1 + \cdots + \sigma_0(y)\, t^s$. Thus, if $\sigma_i(y) \geq 0$ for every $i \in \{1, \ldots, s\}$, then all coefficients of $f(t)$ are non-negative, while the coefficient at t^s is equal to one. It follows that $f(t)$ cannot have strictly positive roots. Hence $y_i \geq 0$ for all $i \in \{1, \ldots, s\}$, which shows the sufficiency of Part I. Now assume that the strict inequality $\sigma_i(y) > 0$ holds for every $i \in \{1, \ldots, s\}$. Then $f(0) = \sigma_s(y) > 0$, i.e., zero is not a root of $f(t)$, and, using the sufficiency of Part I, we arrive a the strict inequalities $y_1 > 0, \ldots, y_s > 0$. This shows the sufficiency in Part II. $\square$

Proposition 4.1 was noticed by Bernig [Ber98, p. 38], who derived it from *Descartes' Rule of Signs*. Our elementary proof (slightly) extends the arguments given in [AH07].

Lemma 4.2. *Let $p_1, \ldots, p_s \in \mathbb{R}[x]$ and $P := (p_1, \ldots, p_s)_{\geq 0}$. Assume that P is non-empty and bounded. Then there exists an algorithm which gets $p_1, \ldots, p_s$ and returns $n(p_1, \ldots, p_s)$.*

Proof. Since P is bounded, we have $n(p_1, \ldots, p_s) \leq 1$. We suggest the following procedure for evaluation of $n(p_1, \ldots, p_s)$.

Procedure: Evaluation of $n(p_1, \ldots, p_s)$

Input: $p_1, \ldots, p_s \in \mathbb{R}[x]$.

Output: $n(p_1, \ldots, p_s)$

1: For $i = 1, \ldots, s$ introduce the formula

$$\Phi_i := \text{"}p_i(x_1, \ldots, x_d) \geq 0\text{"}$$

with free variables $x_1, \ldots, x_d$.

2: Set $n := 1$.

3: Introduce the formula

$$\Phi := \text{"}\prod_{\substack{J \subseteq \{1,\ldots,s\} \\ |J|=n}} \sum_{j \in J} p_j(x_1, \ldots, x_d)^2 = 0\text{"}$$

with free variables $x_1, \ldots, x_d$.

4: Verify whether the sentence

$$\Psi := \text{"}(\exists x_1) \ldots (\exists x_d)\ \Phi \wedge \Phi_1 \wedge \ldots \wedge \Phi_s\text{"}$$

is true or not.

5: If Ψ is true and $n < s$, set $n := n + 1$ and go to Step 3.

6: If Ψ is true and $n = s$, return n and stop.

7: If Ψ is false, set $n := n-1$, return n, and stop

It is not hard to see that the above procedure terminates in a finite number of steps and returns $n(p_1,\dots,p_s)$. $\square$

Proof of Theorem 1.1. As in the previous proofs, we first show the existence of $q_0,\dots,q_n$ from the assertion and then discuss the algorithmic part. We define $q_i,\ 0\le i\le n$, by the formula

$$q_i(x) := \begin{cases} 1-g_{M,\lambda,k}(x) & \text{for } i=0,\\ \sigma_{s-n+i}(p_1(x),\dots,p_s(x)) & \text{for } 1\le i\le n,\end{cases} \tag{4.2}$$

where $k\in\mathbb{N}$, $M\in\mathbb{N}_0$, and $\lambda>0$ will be fixed later. (We recall that $g_{M,\lambda,k}(x)$ is defined by (3.3).) Let us first present a brief sketch of our arguments. It turns out that the polynomials $q_1,\dots,q_n$, which are defined with the help of elementary symmetric functions, represent P locally, that is, P and $(q_1,\dots,q_n)_{\ge 0}$ coincide in a neighborhood of P. In order to pass to the global representation, the additional polynomial q_0 is chosen in such a way that the sublevel set $(q_0)_{\ge 0}$ approximates P sufficiently well.

Given $\varepsilon>0$ let us consider the set $P(M,\varepsilon)$ defined by (3.1). By Theorem 3.2 there exist $M\in\mathbb{N}_0$ and $\varepsilon_0>0$ such that $P(M,\varepsilon_0)$ is bounded. Since $n<s$ it follows that $\sigma_i(p_1(x),\dots,p_s(x))>0$ for all $x\in P$ and $1\le i\le s-n$. Thus, the above strict inequalities hold also for x in a small neighborhood of P. Consequently, by Theorem 3.2, we can fix an $\varepsilon\in(0,\varepsilon_0]$ such that $\sigma_i(p_1(x),\dots,p_s(x))>0$ for all $x\in P(M,\varepsilon)$ and $1\le i\le s-n$. We define the sets

$$Q := \left\{x\in\mathbb{R}^d : q_i(x)\ge 0 \text{ for } 0\le i\le n\right\} \quad\text{and}\quad Q_0 := \left\{x\in\mathbb{R}^d : q_i(x)>0 \text{ for } 0\le i\le n\right\}.$$

Let us consider an arbitrary $x\in P$. Obviously, $q_i(x)\ge 0$ for $1\le i\le n$, where all inequalities are strict if $x\in P_0$. Assume that λ and k satisfy (3.4) and (3.5). Then, by Lemma 3.4, $q_0(x)\ge 0$, where the inequality is strict if $x\in P_0$. Hence $P\subseteq Q$ and $P_0\subseteq Q_0$. Let us show the reverse inclusions. Let $x\in Q_0$. Then, by the definition of $q_0,\dots,q_n$, we have $\sigma_i(p_1(x),\dots,p_s(x))>0$ for $s-n+1\le i\le s$ and $g_{M,\lambda,k}(x)<1$. But, by the choice of ε and $g_{M,\lambda,k}(x)$, we also have $\sigma_i(p_1(x),\dots,p_s(x))>0$ for $1\le i\le s-n$. Thus, $\sigma_i(p_1(x),\dots,p_s(x))>0$ for $1\le i\le s$, and, in view of Proposition 4.1(II), we have $p_i(x)>0$ for $1\le i\le s$. This shows the inclusion $Q_0\subseteq P_0$. The inclusion $Q\subseteq P$ can shown analogously (by means of Proposition 4.1(I)).

Finally we discuss the constructive part of the statement. By Lemma 4.2, n is computable. Consequently, the polynomials $q_1,\dots,q_n$ are also computable, since they are arithmetic expressions in $p_1,\dots,p_s$. The computability of q_0 follows from directly from Theorem 3.2. $\square$

Proof of Theorem 1.2. The polynomials $q_1,\dots,q_i$ will be defined by

$$q_i(x) := \begin{cases} \sigma_{s-n+1}(p_1(x),\dots,p_s(x)) - g_{M,\lambda,k}(x)^l h_\mu(x)^m & \text{for } i=1,\\ \sigma_{s-n+i}(p_1(x),\dots,p_s(x)) & \text{for } 2\le i\le n,\end{cases} \tag{4.3}$$

where $k,l,m\in\mathbb{N},\ M\in\mathbb{N}_0,\ \lambda>0,\ \mu>0$ will be fixed below.

We give a rough description of the arguments. We start with the same remark as in the proof of Theorem 1.1. Namely, polynomials $\sigma_j(p_1(x),\dots,p_s(x))$ with $s-n+1\le j\le s$ represent P locally. We shall disturb the polynomial $\sigma_{s-n+1}(p_1(x),\dots,p_s(x))$ by subtracting an appropriate non-negative polynomial $g_{M,\lambda,k}(x)^l h_\mu(x)^m$ which is small on P, has high order zeros at the points of X, and is large for all points x sufficiently far away from P. See also Fig. 2 for an illustration of Theorem 1.2 in the case $d=2$.

We first show the existence of $q_1,\dots,q_n$ from the assertion. Given $\varepsilon>0$, let us consider the set $P(M,\varepsilon)$ defined by (3.1). By Theorem 3.2 there exist $M\in\mathbb{N}_0$ and $\varepsilon_0>0$ such that $P(M,\varepsilon_0)$ is bounded. Since $n<s$ it follows that $\sigma_i(p_1(x),\dots,p_s(x))>0$ for all $x\in P$ and $1\le i\le s-n$. Thus, the above strict inequalities hold also for x in a small neighborhood of P. Consequently, by Theorem 3.2, we can fix $\varepsilon\in(0,\varepsilon_0/2]$ such that $\sigma_i(p_1(x),\dots,p_s(x))>0$ for all $x\in P(2\varepsilon)$ and $1\le i\le s-n$. Let us borrow the notations from the statements of Theorems 3.2 and 3.5.

We set $q_1 := q$ with $q\in\mathbb{R}[x]$ as in Theorem 3.5. Define the semi-algebraic sets

$$Q=(q_1,\dots,q_n)_{\ge 0} \quad\text{and}\quad Q_0 := (q_1,\dots,q_n)_{>0}.$$

Let us consider an arbitrary $x \in P$. Obviously, $q_i(x) \geq 0$ for $2 \leq i \leq n$, where all inequalities are strict if $x \in P_0$. Furthermore, by Theorem 3.5 we also have $q_1(x) \geq 0$ and this inequality is strict if $x \in P_0$. Thus, we get the inclusions $P \subseteq Q$ and $P_0 \subseteq Q_0$.

It remains to verify the inclusions $Q \subseteq P$ and $Q_0 \subseteq P_0$. Let us consider an arbitrary $x \in \mathbb{R}^d \setminus P_0$, that is, for some $i \in \{1, \ldots, s\}$ one has $p_i(x) \leq 0$. If $x \in P(2\varepsilon) \setminus P_0$, then, by the choice of ε, $\sigma_i(p_1(x), \ldots, p_s(x)) > 0$ for all $1 \leq i \leq s - n$. But, on the other hand, by Proposition 4.1(II), $\sigma_j(p_1(x), \ldots, p_s(x)) \leq 0$ for some $1 \leq j \leq s$. Hence we necessarily have $j > s - n$, and we get that $q_{j+n-s}(x) \leq 0$. Consequently $x \in \mathbb{R}^d \setminus Q_0$. Now assume $x \in P(2\varepsilon) \setminus P$. Then, by Proposition 4.1(I), $\sigma_j(p_1(x), \ldots, p_s(x)) < 0$ for some $1 \leq j \leq s$. But, in the same way as we showed above, we deduce that $j > s - n$. Hence $q_{j+n-s}(x) < 0$, which means that $x \in \mathbb{R}^d \setminus Q$. If $x \in \mathbb{R}^d \setminus P(2\varepsilon)$, then, by Theorem 3.5, one has $q_1(x) < 0$, and by this $x \in \mathbb{R}^d \setminus Q$.

As for the algorithmic part of the assertion, we notice that $n = n(p_1, \ldots, p_s)$ can be easily computed from X. The computability of q_1 follows from Theorem 3.5. $\square$

Remark 4.3. We mention that the "combinatorial component" of our proofs (dealing with elementary symmetric functions) resembles in part the proof of Theorem 1.4. However, the crucial parts of the proofs of Theorems 1.1 and Theorem 1.2 concerning the approximation of P are based on different ideas. The polynomials $q_1, \ldots, q_d$ from Theorem 1.4 can be computed in a rather straightforward way; see [AH07, Section 4]. In contrast to this, the constructive parts of the proofs of Theorems 1.1 and 1.2 use decidability of the first order logic over reals and, by this, lead to algorithms of extremely high complexity. Even though Theorem 2.2 and Proposition 4.1 were also used in [Ber98], our proofs cannot be viewed as extensions of the proofs from [Ber98]. $\square$

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